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A covariant and gauge invariant formulation of the cosmological "backreaction"

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Abstract

Using our recent proposal for defining gauge invariant averages we give a general-covariant formulation of the so-called cosmological "backreaction". Our effective covariant equations allow to describe in an explicitly gauge invariant form the way classical or quantum inhomogeneities affect the average evolution of our Universe.

1 Introduction

It is well known that the homogeneous and isotropic Friedman-Lemaitre-Robertson-Walker (FLRW) metric describes the geometric properties of our Universe only on sufficiently large scales of distance, and has to be interpreted as the "averaged" cosmological metric: namely, the metric emerging from an appropriate smoothing-out of the local inhomogeneities and anisotropies. The same interpretation of averaged variables has to be assigned to the matter energy and momentum density, sourcing the FLRW metric in the cosmological Einstein equations.

A problem (a rather old one, see for instance [1]) thus appears due to the fact that the Einstein equations for the *averaged geometry* are different, in general, from the *averaged Einstein equations*. This is because the averaging procedure does not commute, in general, with the non-linear differential operators appearing in Einstein's equations. As a consequence, the dynamics of the averaged geometry is affected by so-called "backreaction" terms, originating from the contribution of the inhomogeneities present in the metric and matter sectors.

Interest in these themes has considerably risen after the suggestion that the cosmic acceleration, recently observed on large scales, could be unrelated to phantomatic dark-energy sources, but – perhaps more simply – to the dynamical effects of the backreaction (see e.g. [2]) thus solving the well-known"coincidence problem". This possibility has focused current researches on a new problem: the gauge invariance of the averaging procedure. Indeed, in the absence of gauge invariance, the computed backreaction effects depend not only on the hypersurface chosen to compute the average integrals, but also on the chosen coordinate frame (see e.g. [3]-[6] for recent discussions).

At present, the most commonly used averaging procedure in a cosmological context is based on the foliation of space-time into three-dimensional hypersurfaces comoving with the matter sources, and on the volume integration over such spacelike hypersurfaces [7] (see [8] for a recent review, and [6, 9] for an extension to more general hypersurfaces). This procedure is applied, in particular, to the scalar part of the cosmological Einstein equations, i.e. to the so-called Hamiltonian constraint and Raychaudhuri's equation. By spatially averaging such equations on a comoving domain D (and considering, for simplicity, dust fluid sources) one obtains [7]:

$$\left(\frac{\dot{a}_D}{a_D}\right)^2 = \frac{8\pi G}{3}\langle\rho\rangle_D - \frac{1}{6}\left(\langle Q\rangle_D + \langle\mathcal{R}\rangle_{\mathcal{D}}\right), \tag{1.1}$$

$$-\frac{\ddot{a}_D}{a_D} = \frac{4\pi G}{3}\langle\rho\rangle_D - \frac{1}{3}\langle Q\rangle_D. \tag{1.2}$$

Here the dot denotes derivatives with respect to the cosmic time of the comoving synchronous coordinates, a_D is an effective scale factor, related to the volume V_D of the integration domain (normalized by a reference volume scale V_{D_0}) by $a_D = (V_D/V_{D_0})^{1/3}$, where

$$V_D = \int_D d^3x \sqrt{|\det g_{ij}|}, \tag{1.3}$$

and g_{ij} is the intrinsic metric of the comoving hypersurfaces. Also, the brackets denote spatial average over D, namely

$$\langle \cdots \rangle_D = \frac{1}{V_D} \int_D d^3x \sqrt{|\det g_{ij}|}\, (\cdots)\,, \tag{1.4}$$

ρ is the energy density of the dust sources, and $\mathcal{R}$ is the scalar intrinsic curvature associated with the spatial metric g_{ij}. Finally, $\langle Q \rangle$ is a correction called "kinematical" backreaction, arising in a particular gauge (see section 2) from the averages of two scalar quantities: the trace Θ of the expansion tensor and the scalar shear σ^2:

$$\langle Q \rangle_D = \frac{2}{3}\left(\langle \Theta^2 \rangle_D - \langle \Theta \rangle_D^2\right) - 2\langle \sigma^2 \rangle_D. \tag{1.5}$$

The above averaged equations are obtained within a spatial slicing of the space-time manifold induced by the flow lines of the matter sources. In this paper we present a covariant version of the effective equations for the averaged cosmological quantities (called hereafter, for simplicity, averaged cosmological equations), based on a recently proposed gauge invariant averaging prescription [4]: the backreaction effects we obtain depend on the hypersurface chosen to define the physical observer, but *do not depend* on the particular choice of coordinates. In the appropriate class of gauges we recover the recent results given in [6, 9]. In addition, for an observer at rest with respect to the matter sources, we recover the same results as in [7, 8].

The paper is organized as follows. In Sect. 2 we present an explicitly covariant version of the Hamiltonian constraint, Raychaudhuri's equation, and of the projected conservation equation for a generic foliation of spacetime and energy-momentum tensor. In Sect. 3 we briefly summarize the main aspects of our gauge-invariant averaging prescription, and we derive the corresponding general-covariant version of the averaged cosmological equations. Our conclusive remarks are briefly presented in Sect. 4.

2 Covariant ADM equations

In order to introduce spatial averages of physical quantities we consider a general class of foliations of spacetime by spacelike hypersurfaces $\Sigma(A)$ over which a scalar field $A(x)$ takes

constant values (the so-called level-sets of A). Let n_μ be the future-directed unit normal to $\Sigma(A)$, defined by

$$n_\mu = -\frac{\partial_\mu A}{(-\partial_\mu A \partial_\nu A \; g^{\mu\nu})^{1/2}} \quad , \qquad\qquad n_\mu n^\mu = -1 \tag{2.1}$$

(we are using the metric signature $(-,+,+,+)$). Let us also introduce the projector $h_{\mu\nu}$ into the hypersurfaces by:

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \, , \qquad h_{\mu\rho} h^\rho_\nu = h_{\mu\nu} \, , \qquad h_{\mu\nu} n^\mu = 0 \, . \tag{2.2}$$

The Einstein equations $G_{\mu\nu} = T_{\mu\nu}$ (we use units in which $8\pi G = 1$) can then be projected along n^μ and $h^{\mu\nu}$, and give rise to three (sets of) equations that can be chosen in the following form:

$$G_{\mu\nu} n^\mu n^\nu \;=\; T_{\mu\nu} n^\mu n^\nu \equiv \varepsilon \, , \tag{2.3}$$

$$G_{\mu\nu} n^\mu h^\nu_\rho \;=\; T_{\mu\nu} n^\mu h^\nu_\rho \equiv J_\rho \, , \tag{2.4}$$

$$R_{\mu\nu} h^\mu_\rho h^\nu_\sigma \;=\; T_{\mu\nu} h^\mu_\rho h^\nu_\sigma - \frac{1}{2} h_{\rho\sigma} T \equiv \mathcal{S}_{\rho\sigma} - \frac{1}{2} h_{\rho\sigma} T \, . \tag{2.5}$$

They correspond to an explicitly covariant version of the so-called Arnowitt-Deser-Misner (ADM) equations.

It is always possible to make contact with the more conventional ADM formalism by choosing a class of gauges in which the scalar field $A(x)$ is homogeneous. We shall call it the ADM gauges. In such a gauge the normal vector n_μ takes the form:

$$n_\mu = N(-1,0,0,0) \, , \qquad n^\mu = \frac{1}{N}(1, -N^i) \, , \tag{2.6}$$

where N and N^i are, respectively, the so-called lapse function and shift vector. In this gauge:

$$h^\mu_i = \delta^\mu_i \, , \qquad h^0_0 = 0 \, , \qquad h_{ij} = g_{ij} \, , \tag{2.7}$$

where g_{ij} is the induced 3–metric (or first fundamental form) on Σ. The spacetime metric of the foliated spacetime is given by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \quad , \tag{2.8}$$

and its inverse by:

$$\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu = -N^{-2} (\nabla_0 - N^i \nabla_i)^2 + {}^{(3)}g^{ij} \nabla_i \nabla_j \quad . \tag{2.9}$$

where ${}^{(3)}g^{ij}$ is the inverse of the 3×3 metric g_{ij}.

In this class of ADM gauges Eqs. (2.3) and (2.4) reduce to the convential form of the Hamiltonian and momentum constraints, respectively, while Eq. (2.5) generates the

second order evolution equations. In the ADM context such evolution equations are splitted into twice as many equations, those defining the second fundamental form (or extrinsic curvature) K_i^j, and those describing the evolution of K_i^j itself (see, e.g. [8]).

In order to give a covariant and gauge invariant formulation of the cosmological back-reaction we shall make use of the covariant Eqs. (2.3)–(2.5), which lead to an explicitly scalar form of the Hamiltonian constraint and Rachayduri's equation. To this purpose, let us first consider the spacetime flow generated by the timelike vector field n_μ, and define the projected expansion tensor of the flow worldlines as

$$\Theta_{\mu\nu} \equiv h_\mu^\alpha h_\nu^\beta \nabla_\alpha n_\beta = \frac{1}{3} h_{\mu\nu}\Theta + \sigma_{\mu\nu} + \omega_{\mu\nu} \,, \tag{2.10}$$

where

$$\Theta \equiv \nabla_\mu n^\mu, \qquad \sigma_{\mu\nu} \equiv h_\mu^\alpha h_\nu^\beta \left(\nabla_{(\alpha} n_{\beta)} - \frac{1}{3} h_{\alpha\beta} \nabla_\tau n^\tau \right), \quad \omega_{\mu\nu} \equiv h_\mu^\alpha h_\nu^\beta \nabla_{[\alpha} n_{\beta]}, \tag{2.11}$$

are the expansion scalar, the shear tensor and the rotation tensor, respectively. In our case, with n_μ given by Eq. (2.1), we have a zero rotation tensor and we can write

$$\sigma_{\mu\nu} = \Theta_{\mu\nu} - \frac{1}{3} h_{\mu\nu}\Theta \,, \qquad \sigma^2 \equiv \frac{1}{2} \sigma_\nu^\mu \sigma_\mu^\nu = \frac{1}{2} \left(\Theta_\nu^\mu \Theta_\mu^\nu - \frac{1}{3} \Theta^2 \right) \,, \tag{2.12}$$

where σ^2 is the shear scalar.

Consider first the Hamiltonian constraint (2.3), whose left hand side can be decomposed as follows:

$$R_{\mu\nu} n^\mu n^\nu + \frac{1}{2} R = \frac{1}{2} \left(\Theta^2 - \Theta_\nu^\mu \Theta_\mu^\nu \right) + \frac{1}{2} \mathcal{R}_s \,. \tag{2.13}$$

In the ADM gauge the term in brackets on the right hand side can be expressed in terms of the extrinsic curvature as:

$$\Theta^2 - \Theta_\nu^\mu \Theta_\mu^\nu = K^2 - K_j^i K_i^j \,, \tag{2.14}$$

while $\mathcal{R}_s$ is a scalar which, in the ADM gauge, goes over to the intrinsic scalar curvature $\mathcal{R}$ associated with the induced metric g_{ij}. We can thus rewrite the Hamiltonian constraint completely in terms of scalar quantities as:

$$\mathcal{R}_s + \Theta^2 - \Theta_\nu^\mu \Theta_\mu^\nu = \mathcal{R}_s + \frac{2}{3}\Theta^2 - 2\sigma^2 = 2T_{\mu\nu} n^\mu n^\nu = 2\varepsilon \,. \tag{2.15}$$

Coming now to the explicitly scalar form of Rachayduri's equation, we note that it corresponds to the linear combination of Eq. (2.3) and of the trace of Eq. (2.5), leading to:

$$R_{\mu\nu} n^\mu n^\nu = T_{\mu\nu} h^{\mu\nu} - \frac{1}{2} T. \tag{2.16}$$

After some straightforward calculation the above equation takes the form:

$$
\begin{aligned}
-n^\mu \nabla_\mu \Theta &= 2\sigma^2 + \frac{1}{3}\Theta^2 - \nabla^\nu (n^\mu \nabla_\mu n_\nu) + \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) n^\mu n^\nu \\
&= 2\sigma^2 + \frac{1}{3}\Theta^2 - \nabla^\nu (n^\mu \nabla_\mu n_\nu) + \varepsilon + \frac{1}{2} T \,. \qquad (2.17)
\end{aligned}
$$

So far we have made no assumptions on the form of $T_{\mu\nu}$. Let us now restrict ourselves to the case of a perfect fluid with:

$$
T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \qquad (2.18)
$$

where u_μ is the 4-velocity comoving with the fluid, and ρ and p are, respectively, the (scalar) energy density and pressure in the fluid's rest frame (the generalization to several non-interacting fluids is straightforward). We stress that u_μ and n_μ are in general distinct: the former depends on the properties of the matter sources, the latter depends on the choice of the hypersurfaces on which we want to average, hence should be determined by the particular problem at hand. With this model of sources we can then express the basic quantities entering our equations (2.15) and (2.17) as:

$$
\varepsilon = T_{\mu\nu} n^\mu n^\nu = (\rho + p)(u^\mu n_\mu)^2 - p \,, \qquad T = T^\mu_\mu = -\rho + 3p \,. \qquad (2.19)
$$

For later use, we note that the trace T of the energy momentum tensor is unaffected by the possible "tilt" (misalignment) between u_μ and n_μ, while the sources of the covariant ADM Eqs. (2.3)–(2.5) – namely the objects that we may call the ADM energy density ε, the ADM pressure $\pi = \mathcal{S}^\rho_\rho/3$, and the ADM current J_ν – become:

$$
\begin{aligned}
\varepsilon &= \rho - (\rho + p) \left(1 - (u^\mu n_\mu)^2 \right) \,, \qquad (2.20) \\
\pi &= p - \frac{1}{3}(\rho + p) \left(1 - (u^\mu n_\mu)^2 \right) \,, \qquad (2.21) \\
J_\nu &= (\rho + p) u^\beta n_\beta (1 + u^\mu n_\mu) u_\nu. \qquad (2.22)
\end{aligned}
$$

On the other hand, a straightforward calculation in the ADM gauge leads to:

$$
(u^\mu n_\mu)^2 = 1 + {}^{(3)}g^{ij} u_i u_j \geq 1 \,, \qquad (2.23)
$$

meaning that this quantity is always larger than 1. We can thus introduce a "tilt angle" α_T such that $\sinh^2 \alpha_T = (u^\mu n_\mu)^2 - 1$, and we can rewrite ϵ and π in the more convenient form

$$
\varepsilon = \rho + (\rho + p) \sinh^2 \alpha_T \,, \qquad \pi = p + \frac{1}{3}(\rho + p) \sinh^2 \alpha_T \,. \qquad (2.24)
$$

Let us finally consider the condition following from the projected energy-momentum conservation law, $n^\mu \nabla^\mu T_{\mu\nu} = 0$. It is known that such condition is not implied by the

two projected scalar Einstein equations considered above, and has to be added to the set of averaged cosmological equations as an additional independent constraint [7, 8]. In the general context we are considering (with $n_\mu \neq u_\mu$), the projected conservation equation can be written explicitly as follows:

$$u^\mu \partial_\mu \left[(\rho + p)\, u^\rho n_\rho\right] + n^\mu \partial_\mu p + (\rho + p)\left[\nabla_\mu u^\mu u^\rho n_\rho - \Theta^{\mu\nu} u_\mu u_\nu\right] = 0. \tag{2.25}$$

3 Covariant averaged equations

Let us begin from the four dimensional integral of a scalar $S(x)$ as defined in [4]:

$$I(S, \Omega) = \int_{\Omega(x)} d^4x \sqrt{-g(x)}\, S(x) \equiv \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)}\, S(x) W_\Omega(x). \tag{3.1}$$

The integration region $\Omega \subseteq \mathcal{M}_4$ is defined in terms of a suitable scalar window function W_Ω, selecting a region with temporal boundaries determined by the space-like hypersurfaces $\Sigma(A)$ (defined in Sect. 2), and with spatial boundary determined by the coordinate condition $B < r_0$, where B is a (positive) function of the coordinates with space-like gradient $\partial_\mu B$, and r_0 is a positive constant. As we are interested in the variation of the volume averages along the flow lines normal to $\Sigma(A)$, we choose in particular the following window function:

$$W_\Omega(x) = n^\mu \nabla_\mu \theta(A(x) - A_0)\theta(r_0 - B(x)), \tag{3.2}$$

where θ is the Heaviside step function, and n_μ is defined in Eq. (2.1).

As discussed in [4], if $B(x)$ is a scalar function the integral (3.1) is not only a scalar under general coordinate transformations but is also gauge invariant (to all orders): namely, it is invariant under the local field reparametrizations induced by any coordinate change when old and new fields are evaluated at the same space-time position. If B is not a scalar, the spatial boundary can be a source of breaking of covariance and gauge invariance. In [10] we will discuss in more detail such a breaking, and confirm that it goes away in the limit of large spatial volumes (with respect to the typical scale of inhomogeneities) [4]. Using the window function (3.2), the integral (3.1) becomes:

$$I(S, A_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} \delta(A(x) - A_0)(-\partial_\mu A\, \partial^\mu A)^{1/2} \theta(r_0 - B(x))\, S(x)\,. \tag{3.3}$$

Let us now consider the derivative of $I(S, A_0)$ with respect to A_0, a quantity that, like I itself, is covariant and gauge invariant (apart from a possible gauge dependence induced by the spatial boundary):

$$\frac{\partial I(S, A_0)}{\partial A_0} = -\int d^4x \sqrt{-g(x)}\, \delta'\left(A(x) - A_0\right)\left(-\partial_\mu A\, \partial^\mu A\right)^{1/2} \theta(r_0 - B(x))\, S(x)$$

$$= - \int d^4x \sqrt{-g(x)}\, \partial_0 \delta\left(A(x) - A_0\right) \left[\partial_0 A(x)\right]^{-1} (-\partial_\mu A\, \partial^\mu A)^{1/2} \theta(r_0 - B(x))\, S(x). \tag{3.4}$$

Within adapted ADM coordinates, where A is homogeneous, we can always choose those with vanishing shift (i.e. with $g_{00} = -N^2$ and $g_{0i} = 0$). In such coordinates we have:

$$\begin{aligned} \frac{\partial I(S, A_0)}{\partial A_0} &= - \int d^4x \sqrt{-g}\; \partial_0 \delta\left(A(t) - A_0\right) (-g^{00})^{1/2} \theta(r_0 - B(x))\, S(x) \\ &= \int d^4x\, \delta\left(A(t) - A_0\right) \partial_0 \left[\sqrt{-g} (-g^{00})^{1/2} \theta(r_0 - B(x))\, S(x)\right] \\ &= \int d^4x\, \sqrt{|\gamma|} \delta(A(t) - A_0) \Big[\theta(r_0 - B(x))\, (N\Theta\, S + \partial_0 S) - \delta(r_0 - B(x)) S \partial_0 B\Big], \end{aligned} \tag{3.5}$$

where $\gamma = \det g_{ij}$, and we have used that, in these coordinates,

$$\Theta = N^{-1} \partial_0 \log \sqrt{\gamma}. \tag{3.6}$$

The above equation can be easily recast into the following covariant and gauge invariant form,

$$\frac{\partial I(S, A_0)}{\partial A_0} = I\left(\frac{\partial_\mu A \partial^\mu S}{\partial_\mu A \partial^\mu A}, A_0\right) + I\left(\frac{S\,\Theta}{(-\partial_\mu A \partial^\mu A)^{1/2}}, A_0\right) - 2I\left(\frac{\partial_\mu A \partial^\mu B}{\partial_\mu A \partial^\mu A} S \delta(r_0 - B), A_0\right), \tag{3.7}$$

that reduces to Eq. (3.5) in the special coordinates we have been using. Note that the last factor of the above equation is absent if n^μ is orthogonal to the gradient of B ($n^\mu \partial_\mu B = 0$), namely if B does not depend on the time coordinate of the gauge used in Eq. (3.5). We shall restrict ourselves to this case hereafter.

Let us define now the covariant averaging prescription for a scalar $S(x)$ on the hypersurfaces of constant A, following [4], as:

$$\langle S \rangle_{A_0} = \frac{I(S, A_0)}{I(1, A_0)}. \tag{3.8}$$

Using Eq. (3.5) we can easily obtain the derivative of the averaged scalar $\langle S \rangle_{A_0}$ as:

$$\frac{\partial \langle S \rangle_{A_0}}{\partial A_0} = \langle \partial_{A_0} S \rangle_{A_0} + \left\langle S \frac{N\Theta}{\partial_0 A} \right\rangle_{A_0} - \langle S \rangle_{A_0} \left\langle \frac{N\Theta}{\partial_0 A} \right\rangle_{A_0}, \tag{3.9}$$

and Eq. (3.7) immediately gives us the generally covariant version of the above equation:

$$\frac{\partial \langle S \rangle_{A_0}}{\partial A_0} = \left\langle \frac{\partial_\mu A \partial^\mu S}{\partial_\mu A \partial^\mu A} \right\rangle_{A_0} + \left\langle \frac{S\,\Theta}{(-\partial_\mu A \partial^\mu A)^{1/2}} \right\rangle_{A_0} - \langle S \rangle_{A_0} \left\langle \frac{\Theta}{(-\partial_\mu A \partial^\mu A)^{1/2}} \right\rangle_{A_0}. \tag{3.10}$$

This is the covariant and gauge invariant generalization of the Buchert-Ehlers commutation rule [11], and will be the starting point for our generalization of the averaged cosmological

equations. However, let us first illustrate the precise connection to the special version of this rule obtained in [11].

In ADM coordinates, with A homogeneous, Eq. (3.3) reads:

$$I(S,A_0)=\int_{\Sigma_{A_0}} d^3x\sqrt{|\gamma(t_0,\vec{x})|}\,S(t_0,\vec{x})\theta\left(r_0-B(t_0,\vec{x})\right)\,, \tag{3.11}$$

where we have called t_0 the time when $A(t)$ takes the constant values A_0, and the averages are referred to a section of the three-dimensional hypersurface Σ_{A_0} where $A(x)=A_0$. This is exactly the type of spatial integrals used in [7] (with a domain D determined by the condition $B(x)<r_0$), and thus leads to the same definition of averages (see Eq. (1.4)). Let us compute, in this case, the corresponding version of Eq. (3.10). Multiplying both sides by $\partial_t A_0$ we obtain the equation

$$\begin{aligned}\partial_t\langle S\rangle_{A_0} &= \left\langle \partial_t S-N^i\partial_i S\right\rangle_{A_0}+\langle SN\Theta\rangle_{A_0}-\langle S\rangle_{A_0}\langle N\Theta\rangle_{A_0}\\ &= \langle Nn^\mu\partial_\mu S\rangle_{A_0}+\langle SN\Theta\rangle_{A_0}-\langle S\rangle_{A_0}\langle N\Theta\rangle_{A_0}\,,\end{aligned} \tag{3.12}$$

which generalises the commutation rule given in [11] to the case of non-vanishing shift vector. For $N^i=0$ the standard result is recovered.

Let us come now to the covariant formulation of the averaged cosmological equations. Starting from the generally covariant volume integral

$$I(1,A_0)=\int d^4x\sqrt{-g}\,\delta(A(x)-A_0)\sqrt{-\partial_\mu A\partial^\mu A}\,\theta(r_0-B(x)) \tag{3.13}$$

we define, along the lines of the previous approach [7], an effective scale factor $\widetilde{a}$ such that

$$\frac{1}{\widetilde{a}}\frac{\partial\,\widetilde{a}}{\partial A_0}=\frac{1}{3I(1,A_0)}\frac{\partial\,I(1,A_0)}{\partial A_0}. \tag{3.14}$$

We then find, using Eqs. (3.7), (3.8),

$$\frac{1}{\widetilde{a}}\frac{\partial\,\widetilde{a}}{\partial A_0}=\frac{1}{3}\left\langle\frac{\Theta}{(-\partial_\mu A\partial^\mu A)^{1/2}}\right\rangle_{A_0}. \tag{3.15}$$

We are now in the position of presenting the covariant generalization of the averaged equation (1.1). Taking the square of the previous equation, using the Hamiltonian constraint in the form of Eq. (2.15), and explicitly reintroducing Newton's constant in the formulae, we easily obtain:

$$\begin{aligned}\left(\frac{1}{\widetilde{a}}\frac{\partial\,\widetilde{a}}{\partial A_0}\right)^2 &= \frac{8\pi G}{3}\left\langle\frac{\varepsilon}{(-\partial_\mu A\partial^\mu A)}\right\rangle_{A_0}-\frac{1}{6}\left\langle\frac{\mathcal{R}_s}{(-\partial_\mu A\partial^\mu A)}\right\rangle_{A_0}\\ &-\frac{1}{9}\left[\left\langle\frac{\Theta^2}{(-\partial_\mu A\partial^\mu A)}\right\rangle_{A_0}-\left\langle\frac{\Theta}{(-\partial_\mu A\partial^\mu A)^{1/2}}\right\rangle_{A_0}^2\right]+\frac{1}{3}\left\langle\frac{\sigma^2}{(-\partial_\mu A\partial^\mu A)}\right\rangle_{A_0}.\end{aligned} \tag{3.16}$$

In order to arrive at the covariant generalization of the second Buchert's equation (1.2) we start with the simple relation

$$\frac{1}{\widetilde{a}}\frac{\partial^2\widetilde{a}}{\partial A_0^2}=\frac{\partial}{\partial A_0}\left(\frac{1}{\widetilde{a}}\frac{\partial\widetilde{a}}{\partial A_0}\right)+\left(\frac{1}{\widetilde{a}}\frac{\partial\widetilde{a}}{\partial A_0}\right)^2. \tag{3.17}$$

Using Eq. (3.15), and the general commutation rule (3.10) for the first term on the right hand side, we obtain

$$\begin{aligned}-\frac{1}{\widetilde{a}}\frac{\partial^2\,\widetilde{a}}{\partial A_0^2} \quad = \quad & \frac{2}{9}\left\langle\frac{\Theta}{(-\partial_\mu A\partial^\mu A)^{1/2}}\right\rangle_{A_0}^2+\frac{1}{6}\left\langle\frac{\partial_\mu A\partial^\mu(\partial_\nu A\partial^\nu A)}{(-\partial_\mu A\partial^\mu A)^{5/2}}\Theta\right\rangle_{A_0}\\ & -\frac{1}{3}\left\langle\frac{\Theta^2}{(-\partial_\mu A\partial^\mu A)}\right\rangle_{A_0}+\frac{1}{3}\left\langle\frac{\partial_\mu A\partial^\mu\Theta}{(-\partial_\mu A\partial^\mu A)^{3/2}}\right\rangle_{A_0}.\end{aligned} \tag{3.18}$$

Inserting then the covariant Raychaudhuri's equation (2.17) in the last term of the above equation we are lead to the final result,

$$\begin{aligned}-\frac{1}{\widetilde{a}}\frac{\partial^2\,\widetilde{a}}{\partial A_0^2} \quad = \quad & \frac{4\pi G}{3}\left\langle\frac{\varepsilon+3\pi}{(-\partial_\mu A\partial^\mu A)}\right\rangle_{A_0}-\frac{1}{3}\left\langle\frac{\nabla^\nu(n^\mu\nabla_\mu n_\nu)}{(-\partial_\mu A\partial^\mu A)}\right\rangle_{A_0}+\frac{1}{6}\left\langle\frac{\partial_\mu A\partial^\mu(\partial_\nu A\partial^\nu A)}{(-\partial_\mu A\partial^\mu A)^{5/2}}\Theta\right\rangle_{A_0}\\ & -\frac{2}{9}\left[\left\langle\frac{\Theta^2}{(-\partial_\mu A\partial^\mu A)}\right\rangle_{A_0}-\left\langle\frac{\Theta}{(-\partial_\mu A\partial^\mu A)^{1/2}}\right\rangle_{A_0}^2\right]+\frac{2}{3}\left\langle\frac{\sigma^2}{(-\partial_\mu A\partial^\mu A)}\right\rangle_{A_0},\end{aligned} \tag{3.19}$$

where we have used the relation $\rho-3p=\varepsilon-3\pi$ (see Eq. (2.24)). Equations (3.16) and (3.19), together with the averaged conservation equation discussed below, are the main results of this paper.

Let us now observe that Eq. (3.16), written in the ADM gauge, and multiplied by $(\partial_t A_0)^2$, reduces to the equation recently presented in [6, 9]:

$$\begin{aligned}\left(\frac{1}{\widetilde{a}}\frac{\partial\widetilde{a}}{\partial t}\right)^2 \quad = \quad & \frac{8\pi G}{3}\langle N^2\rho\rangle_{A_0}+\frac{8\pi G}{3}\left\langle N^2(\rho+p)\sinh^2\alpha_T\right\rangle_{A_0}-\frac{1}{6}\langle N^2\mathcal{R}\rangle_{A_0}\\ - \quad & \frac{1}{9}\left(\langle N^2\Theta^2\rangle_{A_0}-\langle N\Theta\rangle_{A_0}^2\right)+\frac{1}{3}\langle N^2\sigma^2\rangle_{A_0}.\end{aligned} \tag{3.20}$$

We have used Eq. (2.24) to replace the ADM parameter ε with the fluid proper energy and pressure. When $\sinh^2\alpha_T=0$ (namely when the averaging hypersurfaces coincide with those orthogonal to the fluid velocity, $n_\mu=u_\mu$), we exactly recover the corresponding Buchert's equation (see the second paper of [7]). In the synchronous gauge ($N=1$) and for dust sources ($p=0$) we recover instead Buchert's Eq. (1.1), apart from the additional contribution arising from a nonvaninshing tilt angle α_T.

Consider now the second equation (3.19), and let us first note that

$$\frac{1}{\widetilde{a}}\frac{\partial^2\,\widetilde{a}}{\partial t^2}=\left(\frac{\partial A_0}{\partial t}\right)^2\frac{1}{\widetilde{a}}\frac{\partial^2\,\widetilde{a}}{\partial A_0^2}+\frac{1}{\widetilde{a}}\frac{\partial^2\,A_0}{\partial t^2}\frac{\partial\widetilde{a}}{\partial A_0}. \tag{3.21}$$

We go then to the ADM coordinates, as before, imposing in addition the convenient gauge choice $N^i = 0$, and we apply Eq. (2.24) to express ε and π in terms of ρ and p. By using Eq. (3.19) to eliminate the first term on the right hand side of Eq. (3.21) we obtain

$$\begin{aligned} -\frac{1}{\tilde{a}}\frac{\partial^2 \tilde{a}}{\partial t^2} &= \frac{4\pi G}{3}\left\langle N^2(\rho+3p)\right\rangle_{A_0} + \frac{8\pi G}{3}\left\langle N^2(\rho+p)\sinh^2\alpha_T\right\rangle_{A_0} \\ &- \frac{2}{9}\left(\langle N^2\Theta^2\rangle_{A_0} - \langle N\Theta\rangle^2_{A_0}\right) + \frac{2}{3}\langle N^2\sigma^2\rangle_{A_0} \\ &- \frac{1}{3}\left\langle \Theta\frac{\partial N}{\partial t}\right\rangle_{A_0} - \frac{1}{3}\left\langle N\ {}^{(3)}g^{ij}\nabla_i\nabla_j N\right\rangle_{A_0}, \end{aligned} \tag{3.22}$$

in agreement with [6]. Again, for $\sinh^2\alpha_T = 0$, we also recover the corresponding Buchert's equation (see for instance [8]). We may note that, when $\sinh^2\alpha_T \neq 0$, the "tilt effects" give a negative contribution (assuming $\rho + p > 0$) to the average cosmic acceleration.

Let us conclude this section by noting that, as anticipated in Sect. 2, the set of averaged cosmological equations has to be complemented by the general-covariant average of the conservation equation (2.25). Such an operation can be performed straightforwardly, according to the general procedure outlined above. We shall present here, for simplicity, the covariant version of the averaged conservation equation in the particular case in which the space-time foliation is referred to the fluid comoving frame. In such a case, setting $n_\mu = u_\mu$ in Eq. (2.25), and applying the general-covariant commutation rule (3.10), we obtain

$$\frac{\partial}{\partial A_0}\langle\rho\rangle_{A_0} = -\left\langle \frac{\Theta\, p}{(-\partial_\mu A\partial^\mu A)^{1/2}}\right\rangle_{A_0} - \langle\rho\rangle\left\langle \frac{\Theta}{(-\partial_\mu A\partial^\mu A)^{1/2}}\right\rangle_{A_0} \tag{3.23}$$

(note that, in this case, the scalar A corresponds to the velocity potential of the fluid sources). Going, as before, to the ADM coordinates, and multiplying by $\partial_t A_0$, we recover the known result

$$\frac{\partial}{\partial t}\langle\rho\rangle_D + \langle N\Theta\, p\rangle_D - \langle\rho\rangle\,\langle N\Theta\rangle_D = 0, \tag{3.24}$$

already presented in various papers [7, 8].

4 Conclusion

The main results of this paper are the covariant and gauge invariant formulation of the Buchert-Ehlers commutation rule, Eq. (3.10), and of the effective equations for the averaged evolution of a perfect fluid-dominated Universe, Eqs. (3.16), (3.19),(3.23). The average is performed over a generic class of hypersurfaces, not necessarily orthogonal to the fluid flow lines. We stress that our results allow to compute averaged quantities in a completely arbitrary coordinate system.

The results obtained in this paper can be directly applied to the case of the quantum cosmological backreaction, by using the correspondence between quantum expectation values and classical averages performed over all three-dimensional space, as illustrated in details in [4]. Hence, in particular, can be applied to study the effect of the quantum fluctuations within the canonical formalism of cosmological perturbation theory.

At the same time, the classical averaged equations (possibly extended to light-like hypersurfaces) may provide a covariant starting point for determining whether present inhomogeneities can significantly contribute to the observed cosmic acceleration.

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